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Critical behaviour of the bond-diluted Potts model on Sierpinski carpets

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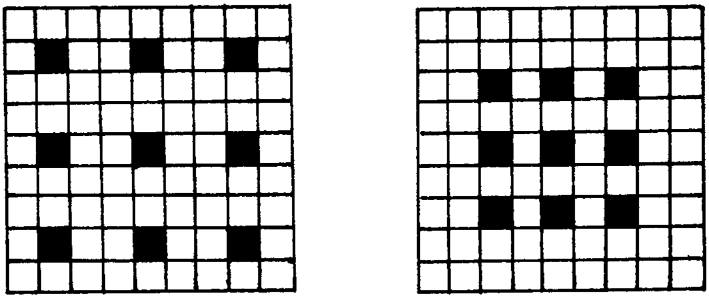
Abstract. A four-parameter bond-moving renormalisation is used to study the critical properties of the bond-diluted q-state Potts model on Sierpinski carpets. Fixed points, critical exponents and a phase diagram are obtained. In our results, there is a borderline value $q = q_t$; for $q < q_t$, the diluted model exhibits the same critical behaviour as the pure system. However, for $q > q_t$ there is crossover to a new diluted fixed point. This behaviour is similar to that of the diluted system on a regular lattice with translational invariance. We give some values of q_t for carpets with different (b, l) where b and l are structure parameters of carpets.

1. Introduction

Many studies of spin systems on fractals have been carried out using various theoretical methods. In particular, Gefen *et al* (1983, 1984a, b) treated the Ising model on three basic types of fractal by using PSRG methods. Their studies showed that the critical properties of fractals depend on some geometrical factors. Very recently Hu (1985), Lin Bin and Yang (1986), Lin Bin (1986), Yang (1987), Qin and Yang (1986) and Ling Hao and Yang (1987) further investigated the same problem and some significant results have been obtained. In this paper, we consider a bond-diluted Potts model on Sierpinski carpets.

We describe the dilution by introducing two concentrations w and p, where w denotes the concentration of occupied bonds which border an eliminated subsquare and p the concentration of all the other occupied bonds, and analyse the critical properties of the diluted system by constructing a four-parameter bond-moving RG. For the problem of bond dilution, it is quite obvious that when the bond concentration is below the percolation concentration, the system consists of an aggregate of finite connected clusters and therefore no phase transition is expected. The percolation concentration can be determined by the fixed point of RG transformation at $T = 0$. In our calculation, the percolation fixed point is a tricritical point (see table 3).

In another respect, for a diluted Potts model on a regular lattice with translational invariance, the critical behaviour of the system is the same as that of the pure system for $q < q_t = 2$ and crosses over a new form for $q > q_t = 2$ (Yeomans and Stinchcombe 1980). Our results showed that similar phenomena will also occur on fractals. Through investigating the carpets with different (b, l), where b and l are the structure parameters of carpets ($l \times l$ subsquares are eliminated from $b \times b$ subsquares (Gefen *et al* 1984b)), we find that, under given b, the q_t decreases as l increases for the carpets with central cutout (see table 4) and the q_t will be equal for some carpets with the same (b, l) but



(a) (b)

Figure 1. (a) and (b) are the first construction stage for Sierpinski carpets with the same b and l ($b=9$, $l=3$) but a different way of cutout. They both have the same $q_t=6.56$.

different ways of cutout because they lead to exactly the same RG recursion relations. For example, figures 1(a) and (b) are two carpets with $b=9$ and $l=3$ but different ways of cutout. They both produce the same RG recursion relations (Ling Hao and Yang 1987). Thus, they give the same $q_t=6.56$. Comparing the values of q_t of $l=0$ and different b (see table 4), which implies the carpet has become a lattice with translational invariance, with the known result $q_t=2$ (Yeomans and Stinchcombe 1980), it may be found that the q_t increases and deviates from the known result with increasing b. For example: with $b=2$ and $l=0$, $q_t=2.95$; with $b=5$ and $l=0$, $q_t=5.22$; and with $b=7$ and $l=0$, $q_t=6.45$. It probably reflects the fact that the larger the b factor, the poorer the result of the method of bond-moving RG.

2. Model

The Potts model on carpets is described by the Hamiltonian

$$-\beta H=\sum_{\langle ij\rangle} K_{ij}\delta_{(\sigma_i,\sigma_j)}+\sum_{\langle mn\rangle} J_{mn}\delta_{(\sigma_m,\sigma_n)} \tag{1}$$

where $\langle ij\rangle$ and $\langle mn\rangle$ mean nearest-neighbour bonds, J_{mn} indicates the 'coupling' via a bond which borders an eliminated subsquare and K_{ij} the 'coupling' via all the other bonds (Gefen *et al* 1984b). To introduce dilution, we allow K_{ij} and J_{mn} to be distributed according to

$$\begin{aligned} P(K_{ij})&=p\delta(K_{ij}-K)+(1-p)\delta(K_{ij}) \\ P(J_{mn})&=w\delta(J_{mn}-J)+(1-w)\delta(J_{mn}) \end{aligned} \tag{2}$$

where p and w are the concentrations of an occupied bond with 'coupling' K_{ij} and J_{mn}, respectively.

3. Recursion relations

We employ the same renormalisation scheme as Gefen *et al* (1984b) used for the Ising model, Migdal–Kadanoff's bond-moving renormalisation, to produce the recursion relations for the bond-diluted Potts model. Following Yeomans and Stinchcombe (1980), it is convenient to introduce functions $t(K)$ and $t(J)$

$$t(K)=\frac{e^K-1}{e^K+q-1} \qquad t(J)=\frac{e^J-1}{e^J+q-1}. \tag{3}$$

Here q is the number of Potts states. Thus, the new coupling K'_{ij} and J'_{mn} for carpets with central cutout are given by

$$\begin{aligned} t'(K'_{ij}) &= f(t) = \prod_{\alpha=1}^{b-l} t_\alpha \prod_{\gamma=1}^{l} t_\gamma \\ t'(J'_{mn}) &= g(t) = \prod_{\nu=1}^{b-l} t_\nu \prod_{\eta=1}^{l} t_\eta \end{aligned} \tag{4}$$

where $t_\alpha = t(K_{\alpha 1}+K_{\alpha 2}+\ldots+K_{\alpha b})$, $t_\gamma = t(K_{\gamma 1}+K_{\gamma 2}+\ldots+K_{\gamma(b-l-1)}+J_{\gamma 1}+J_{\gamma 2})$, $t_\nu = t(K_{\nu 1}+K_{\nu 2}+\ldots+K_{\nu\frac{1}{2}(b-1)}+J_{\nu 1})$, $t_\eta = t(K_{\eta 1}+K_{\eta 2}+\ldots+K_{\eta\frac{1}{2}(b-l)-1}+J_{\eta 1}+J_{\eta 2})$ (see figures 2(a) and (b)) and b and l are the structure parameters of carpets: $l\times l$ subsquares are elimated from $b\times b$ subsquares (Gefen *et al* 1984b).

The renormalised distributions of coupling K'_{ij} and J'_{mn} can be denoted by

$$\begin{aligned} P'(t'(K'_{ij})) &= \int \prod dK_{ij}P(K_{ij}) \prod dJ_{mn}P(J_{mn})\delta(t'(K'_{ij})-f(t)) \\ P'(t'(J'_{mn})) &= \int \prod dK_{ij}P(K_{ij}) \prod dJ_{mn}P(J_{mn})\delta(t'(J'_{mn})-g(t)). \end{aligned} \tag{5}$$

In order to find the recursion relations which we need, we approximate the new distributions $P'(t'(K'_{ij}))$ and $P'(t'(J'_{mn}))$ by

$$\begin{aligned} P_{\text{approx}}(t'(K'_{ij})) &= p'\delta(t'(K'_{ij})-t'(K'))+(1-p')\delta(t'(K'_{ij})) \\ P_{\text{approx}}(t'(J'_{mn})) &= w'\delta(t'(J'_{mn}))-t'(J'))+(1-w')\delta(t'(J'_{mn})) \end{aligned} \tag{6}$$

where $t'(K')$, p' and $t'(J')$, w' are determined by setting the first two moments of $P_{\text{approx}}(t'(K'_{ij}))$ and $P_{\text{approx}}(t'(J'_{mn}))$ to be equal to those of the $P'(t'(K'_{ij}))$ and $P'(t'(J'_{mn}))$ (Yeomans and Stinchcombe 1980). This leads to the recursion relations

$$\begin{aligned} t'_{1,2} &= \langle t^2_{1,2}\rangle / \langle t_{1,2}\rangle \\ p'_{1,2} &= (\langle t_{1,2}\rangle)^2 / \langle t^2_{1,2}\rangle. \end{aligned} \tag{7}$$

Here $t_1 = t(K_{ij})$, $t = t(J_{mn})$, $p_1 = p$ and $p_2 = w$ and $\langle\ldots\rangle$ means an average over the renormalised distributions $P'(t')$ of (5).

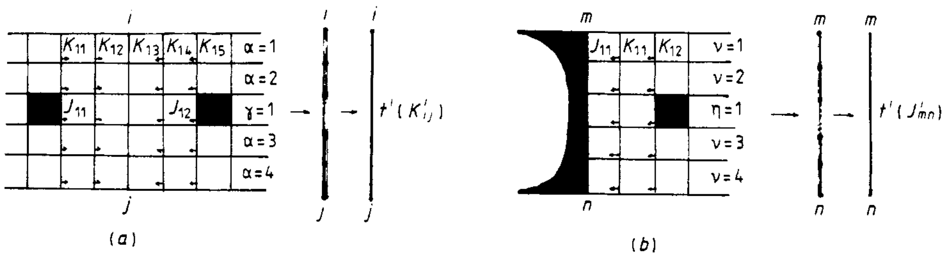


Figure 2. (a) Three steps in obtaining the renormalised $t'(K'_{ij})$ between i and j for the $b=5$ and $l=1$. (b) As (a) for $t'(J'_{mn})$.

4. Results

The numerical results of critical points and critical exponents for $b = 5$ and $l = 1$ are listed in tables 1, 2 and 3. For convenient analysis, q, the number of Potts states, is extended to a continuous variation and only one irrelevant exponent which corresponds to the largest irrelevant eigenvalue is given in the tables.

We see that in table 1 the pure fixed point $F(t^*(K), t^*(J), p^* = w^* = 1)$ is stable for $q < q_t = 4.59$, showing that the critical properties of the system are unchanged by dilution. However, for $q > q_t = 4.59$, the diluted fixed point $F'(t_\alpha^*(K), t_\alpha^*(J), p_\alpha^*, w_\alpha^*)$ appears and the pure fixed point F becomes a tricritical one. The system exhibits crossover to a new critical region dominated by the diluted fixed point F'. The transition remains second order, but has new values of the critical exponents. The crossover exponent

$$\phi = y_2/y_1$$

Table 1. Fixed points for the bond-diluted Potts model on Sierpinski carpets with $b = 5$ and $l = 1$.

q	3	4	4.58	4.59	4.60	5	6
F $t^*(K)$	0.164 92	0.140 21	0.129 60	0.129 44	0.129 27	0.123 07	0.110 33
$t^*(J)$	0.002 30	0.001 22	0.000 89	0.000 886	0.000 88	0.000 73	0.000 47
p^*	1	1	1	1	1	1	1
w^*	1	1	1	1	1	1	1
y_1	0.374 77	0.397 96	0.409 24	0.409 22	0.409 40	0.417 10	0.430 41
y_2	−0.093 40	−0.028 89	−0.000 33	0	0.000 58	0.016 68	0.055 51
ϕ					0.001 42	0.039 99	0.128 97
F' $t_\alpha^*(K)$				0.130 68	0.132 18	0.206 87	0.303 27
$t_\alpha^*(J)$				0.000 92	0.000 96	0.005 03	0.020 68
P_α^*				0.992 84	0.983 47	0.698 29	0.534 35
w_α^*				0.976 26	0.945 81	0.290 38	0.115 48
y_{α_1}				0.408 27	0.407 18	0.363 87	0.330 79
y_{α_2}				−0.000 26	−0.000 33	−0.022 74	−0.071 75

Table 2. Fixed point E and its exponents.

q	$t^*(K)$	$t^*(J)$	p^*	w^*	$\bar{y}_1$	$\bar{y}_2$	$\bar{y}_3$	$\bar{y}_4$
3	0.114 75	1	1	1	0.374 26	0.266 38	0.229 43	−0.132 03
4	0.098 06	1	1	1	0.397 88	0.288 36	0.208 93	−0.068 82
5	0.086 38	1	1	1	0.416 10	0.303 93	0.192 54	−0.021 31
6	0.077 53	1	1	1	0.431 78	0.315 94	0.179 21	−0.013 32

Table 3. Percolation fixed point for carpets with $b = 5$ and $l = 1$.

$t_p^*(K)$	$t_p^*(J)$	p_p^*	w_p^*	y_{p_1}	y_{p_2}	y_{p_3}
1	1	0.2860	0.019 62	0.289 61	0.279 61	−0.370 94

describing the crossover of the system from pure to diluted critical behaviour is listed in table 1 for $q > q_t$. The existence of q_t on a fractal lattice is very similar to Yeomans and Stinchcombe's finding (1980) on the translational lattice. A possible explanation for our finding is that the sign of the specific heat exponent α is changed as q transforms from $q < q_t$ to $q > q_t$. This insight depends on the existence of an extended Harris criterion which can perhaps provide a possible basis for our interpretation. Unfortunately, such an extended criterion, which not only applies to a translational lattice but also to a fractal lattice, is absent as yet.

The percolation fixed point (p_p^*, w_p^*, $t_p^*(K) = t_p^*(J) = 1$) and exponents are shown in table 3. It is a tricritical point, $y_{p_1}, y_{p_2} > 0$ and $y_{p_3} < 0$. Another non-trivial fixed point E and its exponents are listed in table 2. In our four-dimensional space, it is not a marginal (Lin Bin 1986), but a multicritical point. Several trivial fixed points such as $(t^*(K) = t^*(J) = p^* = w^* = 1)$, $(t^*(K) = t^*(J) = p^* = w^* = 0)$, etc, are not listed in the table. The values of q_t for carpets with different (b, l) are shown in table 4. The values of fixed point and exponents obtained for the pure fixed points F and E are in agreement with the known results (Lin Bin 1986).

Because the parameter space is four dimensional, it is impossible to make a complete flow diagram. Thus, we give the flow diagram in a special subspace. Figure 3 shows

Table 4. The values of the q_t of carpets with central cutout for different (b, l).

b	7			5			2
l	3	1	0	3	1	0	0
q_t	4.05	6.11	6.45	2.44	4.59	5.22	2.95

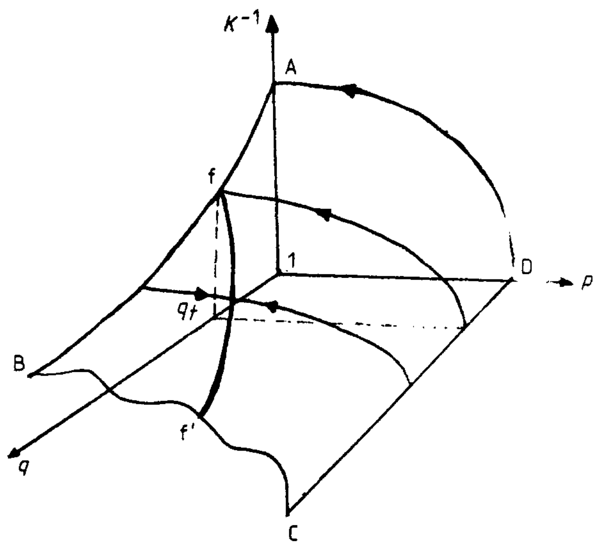


Figure 3. Flows diagram of the diluted Potts model in the subspace of (K^{-1}, p, q). The critical line within the plane of constant q (for example, AD) is the projection of the critical line, which links the percolation fixed point with the pure fixed point, onto the subspace of (K^{-1}, p).

the topology of the RG flow diagram in the subspace of K^{-1}, p and number of states q. The flow diagram in subspace (J^{-1}, w, q) is similar to figure 2. The surface ABCD separates the ferromagnetic and disordered phase. Within this surface there is a line of diluted fixed points, ff′, which meets at point f with line AB at $q = q_t$. The intersection AB of the surface with the plane $p = 1$ corresponding to the pure Potts problem is the pure fixed point line which defines the critical temperature $K_c^{-1}(q)$ of the Potts model. DC is the line of percolation fixed points. The RG trajectories lie in planes of constant q. Thus, flows that begin at the percolation fixed point line are attracted by the pure fixed line for $q < q_t$. Increasing the number of Potts states beyond $q = q_t$, flows that start at the percolation fixed point line and the pure fixed point line go towards the diluted fixed point line as the renormalisation is iterated.

In summary, we constructed a four-parameter bond-moving RG and used it to study the critical properties of the bond-diluted Potts model on Sierpinski carpets. Fixed points, critical exponents and a phase diagram are presented. We found that there is a q_t, such that for $q < q_t$, the critical properties of the system are described by the pure fixed point F which is the critical point of the pure system. However, for $q > q_t$, a new type of critical behaviour dominated by the diluted fixed point F' appears. This situation is similar to that of the diluted Potts problem on lattices with translational invariance.

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